Sheet 5

① Classical Harnack Inequality: if $u : B(0,r) \to \mathbb{R}$ is nonnegative and harmonic, then

$$(*) \quad \frac{r-|x|}{(r+|x|)^{n-1}} r^{n-2} u(0) \le u(x) \le \frac{r+|x|}{(r-|x|)^{n-1}} r^{n-2} u(0) \quad \forall x \in B(0,r)$$

Proof: By Theorem 22 (lectures), since u is harmonic we have

$$u(x) = \int_{\partial B(0,r)} K_r(x,y)\, u(y)\, dS(y) \qquad \forall x \in B(0,r)$$

where $K_r(x,y) = \frac{1}{r\omega_n} \frac{r^2 - |x|^2}{|y-x|^n}$ $\quad x \in B(0,r)$, $y \in \partial B(0,r)$

is the Poisson kernel.

Now note that $\cancel{\frac{r^2-|x|^2}{|y-x|^n}}$ $\quad (r-|x|)^n \le |y-x|^n \le (r+|x|)^n$ since $|y| = r$.

and $\quad r^2 - |x|^2 = (r+|x|)(r-|x|)$

Hence

$$\frac{1}{r\omega_n} \frac{(r+|x|)(r-|x|)}{(r+|x|)^n} \le K_r(x,y) \le \frac{1}{r\omega_n} \frac{(r+|x|)(r-|x|)}{(r-|x|)^n}$$

Hence

$$\frac{1}{r\omega_n} \frac{(r-|x|)}{(r+|x|)^{n-1}} \int_{\partial B(0,r)} u(y)\, dS(y) \le u(x) \le \frac{1}{r\omega_n} \frac{(r+|x|)}{(r-|x|)^{n-1}} \int_{\partial B(0,r)} u(y)\, dS(y)$$

Now use mean value property of u: $\int_{\partial B(0,r)} u(y)\, dS(y) = r^{n-1} \omega_n u(0)$
$\ge 0 \quad (u \ge 0)$.

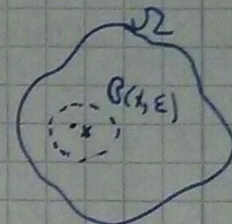
(*) follows.

② Φ - fundamental solution of Laplace's equation

$$\Phi(z) := \begin{cases} -\frac{1}{2\pi} \log|z| & (n=2) \\ \frac{1}{n(n-2)\alpha_n} \frac{1}{|z|^{n-2}} & (n \geq 3) \end{cases} \qquad z \in \mathbb{R}^n,\ z \neq 0.$$

Then $\nabla \Phi(z) = \frac{-1}{\omega_n} \frac{z}{|z|^n}$, $\Delta \Phi(z) = 0$ when $z \neq 0$.

Let $x \in \Omega$ ($\Omega \subset \mathbb{R}^n$ open, bounded, C^1 boundary). Let $\varepsilon > 0$ with $B(x,\varepsilon) \subset\subset \Omega$
and define, ~~for $y \in \Omega \setminus \overline{B(x,\varepsilon)}$~~,



~~$v_\varepsilon(y) :=$~~

$$v_\varepsilon(x) := \int_{\Omega \setminus \overline{B(x,\varepsilon)}} \Phi(y-x)\, dy$$

Then (using similar argument to Problem Sheet 2 (the integral is on a region where Φ is uniformly continuous, so we may differentiate inside the integral)

$$\Delta v_\varepsilon(x) = \int_{\Omega \setminus \overline{B(x,\varepsilon)}} \underbrace{\Delta_y \Phi(y-x)}_{=0}\, dy = 0.$$

But also
$$\int_{\Omega \setminus \overline{B(x,\varepsilon)}} \Delta \Phi(y-x)\, dy = \int_{\partial\Omega \cup \partial B(x,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) \quad (=0)$$

(see tutorial sheet 2 Q1) $\qquad \nabla\Phi(y-x) \cdot \frac{-(y-x)}{|y-x|}$ inward normal!

Hence
$$\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) = -\int_{\partial B(x,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y)$$

$$= -\frac{1}{\omega_n} \int_{\partial B(x,\varepsilon)} \frac{(y-x)\cdot(y-x)}{|y-x|^{n+1}}\, dS(y)$$

$$= \frac{1}{\omega_n} \int_{\partial B(x,\varepsilon)} \frac{1}{|y-x|^{n-1}} \, dS(y) = -\frac{1}{\omega_n} \int_{\partial B(0,\varepsilon)} \frac{1}{|z|^{n-1}} \, dS(z)$$

$$= -\frac{1}{\varepsilon^{n-1}\omega_n} \int_{\partial B(0,\varepsilon)} dS(z) = -1.$$

(c) See later

(3) $\Omega \subset \mathbb{R}^n$ open, bounded $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \qquad f \in C(\bar{\Omega}),\ g \in C(\partial\Omega).$$

Show (Max principle):

$$\max_{x \in \bar{\Omega}} |u(x)| \leq C\left(\max_{x \in \partial\Omega} |g(x)| + \max_{x \in \bar{\Omega}} |f(x)| \right)$$

Proof: Let $\lambda := \max_{x \in \bar{\Omega}} |f(x)|$

Then note $-\Delta\left(u(x) + \frac{|x|^2}{2n}\lambda\right) = -\Delta u(x) - \lambda = -(f(x) + \lambda) \leq 0.$

So $u(x) + \frac{|x|^2}{2n}\lambda$ is subharmonic

Last week: if $v \in \cancel{C^2(\bar{\Omega})}$ subharmonic $\max_{x \in \bar{\Omega}} v(x) = \max_{x \in \partial\Omega} v(x)$

But also, $\max_{x \in \Omega} |v(x)| = \max_{x \in \partial\Omega} |v(x)|$

[Why? Let $M = \max_{\Omega} |v(x)|$. Then if $\exists x_0 \in \Omega$ with $|v(x_0)| = M$, mean value property gives $|v(x_0)| \leq \fint_{B(x_0,r)} |v(y)| \, dy \leq M$ for $B(x_0,r) \subset \Omega$

So $|v(y)| = M \ \forall y \in B(x_0, r)$, so $|v|^{-1}(M)$ open and closed etc.)

Thus we have:

$$\max_{x\in\Omega} |u(x)| = \max_{\bar\Omega} \left| u(x) + \frac{|x|^2}{2n}\lambda - \frac{|x|^2\lambda}{2n} \right|$$

$$\leq \max_{\bar\Omega} \left(\left| u(x) + \frac{|x|^2}{2n}\lambda \right| + \left| \frac{|x|^2\lambda}{2n} \right| \right)$$

$$\leq \max_{\bar\Omega} \underbrace{\left| u(x) + \frac{|x|^2}{2n}\lambda \right|}_{\text{subharmonic}} + \max_{\bar\Omega} \left| \frac{|x|^2\lambda}{2n} \right|$$

$$= \max_{\partial\Omega} \left| u(x) + \frac{|x|^2}{2n}\lambda \right| + \max_{\bar\Omega} \left(\frac{|x|^2\lambda}{2n} \right)$$

$$\leq \max_{\partial\Omega} |u(x)| + \max_{\partial\Omega} \frac{|x|^2\lambda}{2n} + \max_{\bar\Omega} \left| \frac{|x|^2\lambda}{2n} \right|$$

$$= \max_{\partial\Omega} |g(x)| + C\lambda \qquad \left(* \; C \text{ depends on } \max_{x\in\bar\Omega} |x|^2 \text{ ie } \Omega. \right)$$

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(b) If $\begin{cases} -\Delta u_i = f_i & \text{in } \Omega \\ u_i = g_i & \text{on } \partial\Omega \end{cases}$ for $i = 1, 2$ (u_i, f_i, g_i as in (*))

then $\begin{cases} -\Delta(u_1 - u_2) = f_1 - f_2 & \text{in } \Omega \\ u_1 - u_2 = g_1 - g_2 & \text{on } \partial\Omega \end{cases}$

Apply (a) to $u_1 - u_2$:

$$\max_{\bar\Omega} |u_1(x) - u_2(x)| \leq C \left(\max_{\partial\Omega} |g_1(x) - g_2(x)| + \max_{\bar\Omega} |f_1(x) - f_2(x)| \right)$$

ie $\|u_1 - u_2\|_{L^\infty(\Omega)} \leq C \left(\|g_1 - g_2\|_{L^\infty(\partial\Omega)} + \|f_1 - f_2\|_{L^\infty(\Omega)} \right)$

$C(\bar\Omega)$ - u_i cont.

② (c) $K(x,y) := \frac{2x_n}{\omega_n}\frac{1}{|x-y|^n}$ $\quad x\in\mathbb{R}^n_+ \quad y\in\partial\mathbb{R}^n_+ \; (\cong \mathbb{R}^{n-1})$

<u>Show:</u> $\int_{\partial\mathbb{R}^n_+} K(x,y)\,dy = 1 , \quad x\in\mathbb{R}^n_+$

<u>Proof:</u> Key trick: $\int_0^\infty \frac{1}{1+t^2}\,dt = \frac{\pi}{2}$

$$\int_{\mathbb{R}^n_+} K(x,y)\,dy = \frac{2}{\omega_n}\int_{\mathbb{R}^n_+}\frac{x_n}{(|\hat{x}-y|^2+x_n^2)^{n/2}}\,dy \qquad \hat{x}=(x_1,\dots,x_{n-1},0)$$

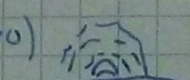
change of variables
$$= \frac{2}{\omega_n}\int_{\underset{\cong\mathbb{R}^{n-1}}{\mathbb{R}^n_+}}\frac{x_n}{(|y|^2+x_n^2)^{n/2}}$$

trick
$$= \frac{2}{\omega_n}\cdot\frac{2}{\pi}\int_{x_n=0}^\infty \frac{1}{1+x_n^2}\int_{y\in\mathbb{R}^{n-1}}\frac{x_n}{(|y|^2+x_n^2)^{n/2}}\,dy\,dx_n$$

$$= \frac{4}{\pi\omega_n}\int_{\mathbb{R}^n_+}\frac{x_n}{(1+x_n^2)(|y|^2+x_n^2)^{n/2}}\,d(y,x_n)$$

polar coordinates on half plane
$$= \frac{4}{\pi\omega_n}\int_{r=0}^\infty\int_{\partial B(0,r)^+}\frac{x_n}{(1+x_n^2)(\underbrace{|y|^2+x_n^2}_{=r^2})^{n/2}}\,dS(y,x_n)\,dr$$

← half ball (with $x_n>0$)



$$= \frac{4}{\pi\omega_n}\int_{r=0}^\infty\frac{1}{r^n}\int_{\partial B(0,r)^+}\frac{x_n}{1+x_n^2}\,dS(y,x_n)\,dr$$

Substitute $rz = (y,x_n)$
$$= \frac{4}{\pi\omega_n}\int_0^\infty\frac{1}{r^n}\cdot r^{n-1}\int_{\partial B(0,1)^+}\frac{r z_n}{1+(rz_n)^2}\,dS(z)\,dr$$

$$= \frac{4}{\pi \omega_n} \int_{\partial B(0,1)^+} z_n \int_0^\infty \frac{1}{1+(r z_n)^2} \, dr \, dS(z)$$

$$= \frac{4}{\pi \omega_n} \int_{\partial B(0,1)^+} \frac{\pi}{2} \, dS(z)$$

$$= \frac{2}{\omega_n} \cdot \underbrace{\frac{\omega_n}{2}}_{= \mathcal{H}^{n-1}(\partial B^+)} = 1$$